BAULKHAM HILLS HIGH SCHOOL

MATHEMATICS EXTENSION 1 ASSESSMENT

December 2012

Time allowed: 50 minutes
plus 5 minutes reading time

STUDENT NUMBER :____________________

TEACHER'S NAME:____________________

Topics Tested: Locus Problems, Polynomials and Induction

BAULKHAM HILLS HS PERSEVERE

Extension 1 Mathematics

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DIRECTIONS

- Full working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Use black or blue pen only (*not pencils*) to write your solutions.
- No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.
- Approved Maths aids and calculators may be used

Question 1 (8 marks) – Start a new page

a) For each of the following expressions, state whether or not they are polynomials.
If it is a polynomial, state its degree, if it isn't explain why it isn't.

(i) $3^{-1}x - 4x^8 + \sqrt{3}$ **1**

(ii) $4x^7 + \sqrt{x}$ **1**

b) If α, β and γ are the roots of $2x^3 - 6x^2 - 4x + 1 = 0$ find

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(iii) $\alpha\beta\gamma$ **1**

(iv) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$ **1**

(v) $\alpha^2 + \beta^2 + \gamma^2$ **2**

Question 2 (9 marks) – Start a new page

a) (i) Sketch without the use of calculus, the polynomial $P(x) = (1-2x)^2(x+1)^3$ showing x and y intercepts. **2**

(ii) Hence, or otherwise solve the inequation $P(x) > 0$ **2**

b) A polynomial is given by $P(x) = x^3 + ax^2 + x - 18$ **2**
Find the values for a if -24 is the remainder when $P(x)$ is divided by $(x-2)$

c) For all positive integers of n, prove by mathematical induction that **3**

$$\sum_{r=1}^{n}(r-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

Question 3 (8 marks) – Start a new page

a) Write a polynomial with degree 3, which is odd, has a zero of 2 and has a remainder 5 when divided by $x+1$ **3**

b) The roots of the equation $4x^3 + 6x^2 + c = 0$ are α, β and $\alpha\beta$(where c is a non-zero constant)

(i) Show that $\alpha\beta \neq 0$ **1**

(ii) Show that $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = 0$ and hence show that the value of $\alpha + \beta$ is -1 **2**

(iii) Find the value of $\alpha\beta$ **2**

Question 4	**(6 marks) – Start a new page**	
a)	Use mathematical induction to show that the expression $7^n + 5$ is divisble by 6 for all positive integers n.	**3**
b)	Solve $4x^3 - 12x^2 + 11x - 3 = 0$ given that the roots are consecutive terms of an arithmetic series.	**3**
Question 5	**(9 marks) – Start a new page**	
a)	Given point P is a variable point with coordinates $(2p^2 + 1, 4)$ (i) Find the equation of the locus of P. (ii) State any restrictions of the point P.	 **1** **1**
b)	Two points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$, and the line joining PQ is parallel to the line $y = mx$. (i) Show that $p + q = 2m$ (ii) Show that the equation of the normal to the parabola at the point P is $x + py = 2p + p^3$ (iii) Show that the point of intersection of the normals from P and Q is $N(-pq\,(p + q), 2 + p^2 + pq + q^2)$ (iv) Find the equation in Cartesian form of the locus of the point N.	 **1** **2** **2** **2**
	~ End of Exam ~	

XI Soln - Assess Task Dec. 2012

Q1

a) i) Yes - Degree 8 ✓

ii) No - $\sqrt{x} = x^{\frac{1}{2}}$

Polynomials must have powers with positive integers. ✓

b) $2x^3 - 6x^2 - 4x + 1 = 0$

i) $\frac{6}{2} = 3$

ii) $\frac{-4}{2} = -2$

iii) $-\frac{1}{2}$

iv) $\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{3}{(-\frac{1}{2})}$

v) $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$ ✓

$= (-2)^2 - 2(3)$

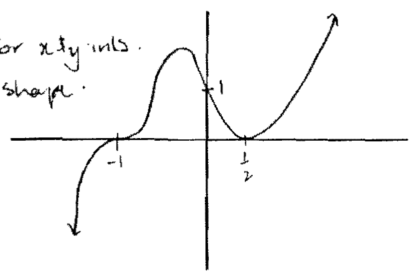
$= 13.$ ✓

Q2

a) i)

✓ 1mk for x+y ints.

✓ 1mk shape.



ii) $x > -1$ and $x \neq \frac{1}{2}$ ✓

b) $-24 = (2)^3 + a(2)^2 + (2) - 18$ ✓

$-16 = 4a$

$a = -4$ ✓

c) $\sum_{r=1}^{n} (r-1)^2 = \frac{n(n-1)(2n-1)}{6}$

ie $0^2 + 1^2 + 2^2 + \ldots (n-1)^2 = \frac{n(n-1)(2n-1)}{6}$

For $n=1$

LHS $= 0^2 = 0$

RHS $= \frac{1(1-1)(2-1)}{6} = 0$

LHS = RHS

$\therefore$ true for $n=1$ ✓

Assume true for $n=k$

ie $0^2 + 1^2 + 2^2 + \ldots + (k-1)^2 = \frac{k(k-1)(2k-1)}{6}$

Prove true for $n=k+1$

Aim: $0^2 + 1^2 + 2^2 + \ldots + (k-1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}$

Proof:

LHS $= 0^2 + 1^2 + 2^2 + \ldots + (k-1)^2 + k^2$

$= \frac{k(k-1)(2k-1)}{6} + k^2$ (by assumption) ✓

$= \frac{k}{6}\left[(k-1)(2k-1) + 6k\right]$

$= \frac{k}{6}(2k^2 - 3k + 1 + 6k)$

$= \frac{k}{6}(2k^2 + 3k + 1)$ ✓

$= \frac{k}{6}(k+1)(2k+1)$

$=$ RHS.

$\therefore$ If it is true for $n=k$ & proven true for $n=k+1$ & since proven true for $n=1$ then it is true for $n=2,3,4,\ldots$

ie all positive integers of n

Note: $P(x) = a(x)(x+2)(x-2)$

$P(-1) = 5 = a(-1)(-1+2)(-1-2)$

$a = \frac{5}{3}$

Q3

a) $P(x) = ax^3 + bx^2 + cx + d$

$P(-x) = -ax^3 + bx^2 - cx + d$

Since $P(x)$ is odd

ie $P(-x) = -P(x)$

$P(-x) + P(x) = 0$

then $b = d = 0$ ✓

$\therefore P(x) = ax^3 + cx$

$P(2) = 0 = 8a + 2c$

$0 = 4a + c$ —①

$P(-1) = 5 = -a - c$ —② ✓

①+② $5 = 3a$

$a = \frac{5}{3}.$

② $\rightarrow$ $5 = -\frac{5}{3} - c$

$c = -\frac{20}{3}$

$\therefore P(x) = \frac{5}{3}x^3 - \frac{20}{3}x$ ✓ 1mk correct form from simult. Solns.

b) $4x^3 + 6x^2 + 0x + c = 0$

i) $\alpha\beta(\alpha\beta) = \frac{-c}{4}$

$\alpha^2\beta^2 = -\frac{c}{4}$

$\alpha\beta = \pm\sqrt{\frac{-c}{4}}$ but $c \neq 0$ ✓

$\therefore \alpha\beta \neq 0$

ii) $\alpha\beta + \alpha(\alpha\beta) + \beta(\alpha\beta) = \frac{0}{4}$ ✓

$\alpha\beta(1 + \alpha + \beta) = 0$

$\alpha\beta \neq 0$ (part i) $1 + \alpha + \beta = 0$

$\alpha + \beta = -1$ ✓

iii) $\alpha + \beta + (\alpha\beta) = \frac{-6}{4}$ ✓

$-1 + \alpha\beta = -\frac{3}{2}$

$\alpha\beta = -\frac{1}{2}.$ ✓

Q4

a) For $n=1$

$7^1 + 5 = 12 = 6 \times 2$, divisible by 6

$\therefore$ true for $n=1$ ✓

Assume true for $n=k$

$7^k + 5 = 6P$

where P is an integer

Investigate for $n=k+1$

Aim: $7^{k+1} + 5 = 6Q$ where Q is an integer.

Proof: LHS $= 7^{k+1} + 5$

$= 7 \times 7^k + 5$

$= 7(6P - 5) + 5$ by assumption ✓

$= 7 \times 6P - 30$

$= 6(7P - 5)$ since $(7P-5)$ is an integer.

$= 6Q$

$\therefore$ divisible by 6 for $n=k+1$ if divisible by 6 for $n=k$

$\therefore$ If it is true for $n=k$, and true for $n=k+1$ and since it is true for $n=1$, then it is true for $n=2,3,\ldots$ ie all positive integers of. ✓

Q4 cont.

b) $4x^3 - 12x^2 + 11x - 3 = 0$

Let the roots be $\alpha - d, \alpha, \alpha + d$

$(\alpha - d) + \alpha + (\alpha + d) = \frac{12}{4}$

$3\alpha = 3$

$\alpha = 1$ ✓

$\alpha(\alpha - d)(\alpha + d) = -\frac{-3}{4}$

but $\alpha = 1 \ldots \Rightarrow (1-d)(1+d) = \frac{3}{4}$

$1 - d^2 = \frac{3}{4}$

$d^2 = \frac{1}{4}$

$d = \pm\frac{1}{2}$ ✓

$\therefore$ roots are $1 - \frac{1}{2}, 1, 1 + \frac{1}{2}$

$\therefore x = \frac{1}{2}, x = 1, x = 1\frac{1}{2}$ ✓

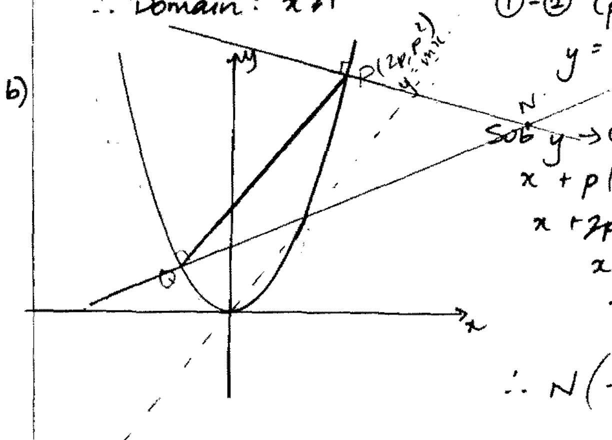
Q5

a) i) $y = 4$ ✓

ii) Since $x = 2p^2 + 1$ ✓

$\therefore$ Domain: $x \geq 1$

b)



i) $m_{PQ} = \frac{p^2 - q^2}{2(p-q)}$

$m = \frac{(p+q)(p-q)}{2(p-q)}$

$2m = p + q$ ✓

ii) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$

$m_T = \frac{2p}{2} = p$.

$\therefore m_N = -\frac{1}{p}$ ✓

Eqn of normal

$y - p^2 = -\frac{1}{p}(x - 2p)$

$-yp + p^3 = x - 2p$. ✓

$x + yp = p^3 + 2p$. —— ①

iii) Normal at Q is $x + qy = 2q + q^3$ —②

①-② $(p-q)y = 2(p-q) + (p^3 - q^3)$ ✓

$y = 2 + p^2 + pq + q^2$

Sub y → ①

$x + p(2 + p^2 + pq + q^2) = p^3 + 2p$.

$x + 2p + p^3 + p^2q + pq^2 = p^3 + 2p$

$x = -p^2q - pq^2$ ✓

$x = -pq(p+q)$

$\therefore N(-pq(p+q), p^2 + pq + q^2 + 2)$

Q5 cont.

iv) $x = -pq(p+q)$

but $p + q = 2m$ ← using this ✓

$x = -pq(2m)$

$pq = \frac{-x}{2m}$.

$y = p^2 + pq + q^2 + 2 \quad + pq - pq$

$y = (p+q)^2 + 2 - pq$

[but $p + q = 2m$ & $pq = \frac{-x}{2m}$] ✓

$y = (2m)^2 + 2 - \left(\frac{-x}{2m}\right)$

$y = 4m^2 + 2 + \frac{x}{2m}$. ✓